ISRAEL JOURNAL OF MATHEMATICS 117 (2000), 183–219

INTRODUCTION TO ℓ_2-METHODS IN TOPOLOGY: REDUCED ℓ_2-HOMOLOGY, HARMONIC CHAINS, ℓ_2-BETTI NUMBERS

BY

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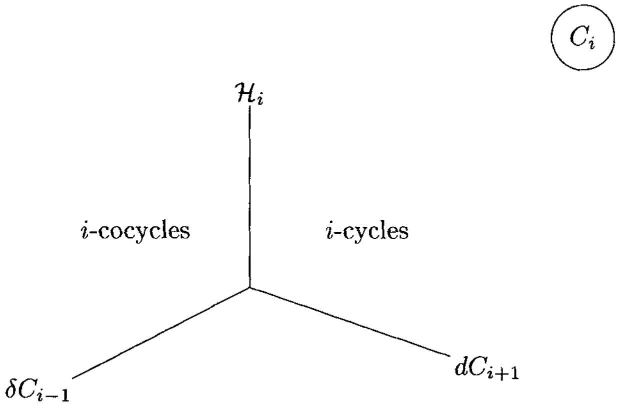
Preface

These are notes from a mini-course at the ETH Zurich addressed to faculty and advanced students. Its purpose was to provide a first acquaintance of the Hilbert space methods in algebraic topology which were initated by Atiyah in 1976 and have become a quite general and important tool during more recent years. Prerequisites are basic algebraic topology of cell-complexes and basic concepts of Hilbert spaces. The definitions (Hilbert-G-module, von Neumann dimension, reduced (co)homology, ℓ_2-Betti numbers of finite complexes) are given, as well as complete proofs of main properties such as homotopy invariance, Poincaré duality, etc. Applications which cannot, or not easily, be done without ℓ_2-Betti numbers concern (partial) Euler characteristic, finitely presented groups, and 4-manifolds; the Cheeger–Gromov lemma on amenable groups is stated and proved. The integrality conjecture known as "Atiyah conjecture" is formulated in a most general way and discussed.

A word about our systematic use of the group of *harmonic* chains, isomorphic to both homology and cohomology groups. To prepare the ground this is illustrated, in a preliminary chapter, by the elementary case of (co-)homology with real coefficients of a finite cell-complex X. The chain groups $C_i(X)$ are finite dimensional vector spaces with a natural scalar product where the cells form an orthonormal basis. Boundary d and coboundary δ are adjoint maps; C_i decomposes into three mutually orthogonal subspaces: dC_{i+1}, δC_{i-1}, and the kernel

* Notes by Guido Mislin, based on lectures by Beno Eckmann, autumn 1997, at the Mathematical Research Institute, ETH Zurich.
Received June 17, 1999

$\mathcal{H}_i$ of the combinatorial Laplacian $\Delta = d\delta + \delta d$ (or equivalently the intersection of the i-cycle and the i-cocycle space), as described in the intuitive picture



Interesting use of the "harmonic" chains $\in \mathcal{H}_i$ representing (co-)homology classes can already be made in that elementary situation.

The ℓ_2-methods appear if a regular covering Y of X is considered, in general an infinite cell-complex, with the covering transformation group G operating freely. The same decomposition, as above, of the Hilbert space of ℓ_2-chains is obtained with the only difference that one has to replace the i-boundary space and the i-coboundary space by their closures. Thus $\mathcal{H}_i$ is isomorphic to *reduced* homology: cycles modulo the closure of the boundaries, and also to reduced cohomology. All these Hilbert spaces admit isometric G-action; they are Hilbert-G-modules, and their von Neumann dimension relative to G, a real non-negative number, plays the role of the vector space dimensions in the finite complex case. In particular, the von Neumann dimension of $\mathcal{H}_i$ is the i-th ℓ_2-Betti number β_i of Y relative to G. If $Y = \tilde{X}$, the universal covering, G the fundamental group of X, it is just called $\beta_i(X)$. For $G = 1$, $Y = X$, one is in the elementary case above. If G is finite, $\beta_i(X)$ is the ordinary Betti number of Y divided by $|G|$. For infinite G the values of the ℓ_2-Betti numbers are more complicated but they nevertheless compute the Euler characteristic exactly as the ordinary Betti numbers do.

Many thanks to Guido Mislin for writing up these notes, very carefully and with many improvements; and to Emmanuel Dror Farjoun for asking us to publish them in this journal despite their introductory character.

Beno Eckmann

1. Finite CW-complexes and $\mathbb{R}$-homology

1.1. Let X be a finite CW-complex with cellular chain complex $(K_*(X), d)$. We write $C_*(X) = \mathbb{R} \otimes K_*(X)$ for the associated chain complex over $\mathbb{R}$. If α_i denotes the number of i-cells of X, then

$$\dim_{\mathbb{R}} C_i(X) = \alpha_i,$$

and $C_i(X)$ has a natural basis $\sigma_1, \ldots, \sigma_{\alpha_i}$ consisting of the i-cells of X. We will consider $C_i(X)$ as a (real) Hilbert space with orthonormal basis $\sigma_1, \ldots, \sigma_{\alpha_i}$ and associated inner product

$$\langle \ , \ \rangle \colon C_i \otimes_{\mathbb{R}} C_i \longrightarrow \mathbb{R}.$$

The boundary operator

$$d_i \colon C_i(X) \longrightarrow C_{i-1}(X)$$

then has an **adjoint**

$$d_i^* = \delta_{i-1} \colon C_{i-1}(X) \longrightarrow C_i(X),$$

given by $\langle \delta_{i-1} x, y \rangle = \langle x, d_i y \rangle$. Thus

$$\begin{aligned} \ker \delta_i &= (\operatorname{im} d_{i+1})^\perp \subset C_i(X), \quad \text{and} \\ \ker d_i &= (\operatorname{im} \delta_{i-1})^\perp \subset C_i(X). \end{aligned}$$

Putting $Z_i = \ker d_i, B_i = \operatorname{im} d_{i+1}, Z^i = \ker \delta_i, B^i = \operatorname{im} \delta_{i-1}$ and $C_i = C_i(X)$, one finds orthogonal decompositions

$$C_i = B^i \perp Z_i = B_i \perp Z^i.$$

Since $\langle \delta_{i-1} x, d_{i+1} y \rangle = 0$ for all $x, y \in C_i$, one has $B^i \perp B_i$ and therefore

$$C_i(X) = B_i \perp B^i \perp (Z_i \cap Z^i),$$

the **Hodge–de Rham decomposition** of $C_i(X)$. The groups

$$\mathcal{H}_i(X) := Z_i(X) \cap Z^i(X)$$

are called the harmonic i-chains of X. One defines the **Laplacian** by

$$\Delta_i = d_{i+1}\delta_i + \delta_{i-1} d_i \colon C_i \longrightarrow C_i.$$

It has the property that

$$\mathcal{H}_i(X) = \{x \in C_i(X) \mid \Delta_i(x) = 0\}.$$

Indeed, it is plain that $Z_i \cap Z^i \subset \ker \Delta_i$. Conversely, if $\Delta_i x = 0$ then

$$d_{i+1}\delta_i x = -\delta_{i-1} d_i x \in B_i \cap B^i = \{0\},$$

thus $\delta_i x \in B^{i+1} \cap Z_{i+1} = \{0\}$ and $d_i x \in B_{i-1} \cap Z^{i-1} = \{0\}$, implying that $x \in Z_i \cap Z^i$.

1.2. The Euler characteristic of X is, as usual, defined by

$$\chi(X) = \sum_i (-1)^i \alpha_i,$$

and we define the Betti numbers by putting

$$b_i(X) = \dim_{\mathbb{R}} \mathcal{H}_i(X).$$

Corollary 1.2.1: *The Euler characteristic of X satisfies*

$$\chi(X) = \sum_i (-1)^i b_i(X).$$

Proof: Since $C_i(X) = B^i \perp B_i \perp \mathcal{H}_i(X)$ and $Z^i = B^i \perp \mathcal{H}_i(X)$, we see that δ_i maps $(Z^i)^\perp \subset C_i(X)$ onto B^{i+1}, inducing an isomorphism

$$B_i \xrightarrow{\cong} B^{i+1} \tag{1}$$

so that $\dim_{\mathbb{R}} B_i = \dim_{\mathbb{R}} B^{i+1}$ for all i. Since $\alpha_i = \dim_{\mathbb{R}} C_i(X)$, it follows then that

$$\chi(X) = \sum_i (-1)^i \dim_{\mathbb{R}} C_i(X) = \sum_i (-1)^i \dim_{\mathbb{R}} \mathcal{H}_i(X) + r,$$

where $r = \sum_i (-1)^i (\dim_{\mathbb{R}} B_i + \dim_{\mathbb{R}} B^i) = 0$. ∎

Similarly, we obtain the following inequalities.

Corollary 1.2.1 (Morse Inequalities): *Let X be a finite CW-complex with α_i i-cells and Betti numbers $b_i, i \in \mathbb{N}$. Then, for every $k \geq 0$,*

$$\alpha_k - \alpha_{k-1} + \alpha_{k-2} - \cdots (-1)^k \alpha_0 \geq b_k - b_{k-1} + b_{k-2} \cdots (-1)^k b_0.$$

Proof: Indeed, using (1), $C_i = B_i \perp B^i \perp \mathcal{H}_i \cong B_i \oplus B_{i-1} \oplus \mathcal{H}_i$ and therefore

$$\sum_{i=0}^{k} (-1)^{k-i} \alpha_i - \sum_{i=0}^{k} (-1)^{k-i} b_i = \dim_{\mathbb{R}} B_k \geq 0. \quad ∎$$

1.3. The cellular $\mathbb{R}$-homology groups are defined by

$$H_i(X;\mathbb{R}) = Z_i(X)/B_i(X).$$

Because $Z_i(X) = B_i(X) \perp \mathcal{H}_i(X)$, the orthogonal projection $Z_i(X) \to \mathcal{H}_i(X)$ induces an isomorphism

$$H_i(X;\mathbb{R}) \xrightarrow{\cong} \mathcal{H}_i(X). \tag{2}$$

In particular, our Betti numbers agree with the usual Betti numbers, $b_i(X) = \dim_{\mathbb{R}} H_i(X;\mathbb{R})$, and they are therefore homotopy invariants.

The real cellular cochain complex $C^* = C^*(X) = \mathrm{Hom}_{\mathbb{R}}(C_*(X), \mathbb{R})$ has differential defined by $\delta^{i-1} = \mathrm{Hom}_{\mathbb{R}}(d_i, \mathbb{R})\colon C^{i-1} \to C^i$. Using the inner product of C_*, one obtains natural isomorphism

$$\Lambda_i\colon C_iX \longrightarrow C^iX, \quad \sigma \mapsto \langle \sigma, \quad \rangle,$$

σ an i-cell of X. Since

$$\begin{aligned}((\Lambda_{i+1}\delta_i)(\sigma)(c) &= \langle \delta_i\sigma, c\rangle = \langle \sigma, d_{i+1}c\rangle \\ &= \big(\Lambda_i(\sigma)\big)(d_{i+1}c) = \big((\delta^i\Lambda_i)(\sigma)\big)(c),\end{aligned}$$

$\Lambda_*\colon (C_*, \delta) \to (C^*, \delta)$ defines an isomorphism of cochain complexes, mapping $Z^i(X)$ isomorphically onto $\ker \delta^i$, and $B^i(X)$ onto $\mathrm{im}\, \delta^{i-1}$. We may therefore refer, under that isomorphism, to the elements of $Z^i(X) \subset C_i(X)$ as **cocycles**, and $B^i(X) \subset C_i(X)$ as **coboundaries**. The harmonic chains are then those chains, which are simultaneously cycles and cocycles.

1.4. A cellular map $f\colon X \to Y$ induces $f_i\colon C_i(X) \to C_i(Y)$ mapping cycles to cycles, but in general, cocycles are not mapped to cocycles (of course, the adjoint f_i^* maps cocycles to cocycles). If we wish to view $\mathcal{H}_i$ as a (co)functor, we may proceed as follows. Using the identification (2): $H_i(X;\mathbb{R}) \to \mathcal{H}_i(X)$ induced by the orthogonal projection $Z_i(X) \to \mathcal{H}_i(X)$, we obtain a **functor** $\mathcal{H}_i$ on the category of finite CW-complexes and cellular maps; the so induced maps

$$f_!\colon \mathcal{H}_i(X) \to \mathcal{H}_i(Y)$$

given by

$$f_!\colon \mathcal{H}_i(X) \xrightarrow{\cong} H_i(X;\mathbb{R}) \xrightarrow{H_i(f)} H_i(Y;\mathbb{R}) \xleftarrow{\cong} \mathcal{H}_i(Y)$$

depend obviously on the homotopy class of f only, showing that $\mathcal{H}_i$ is a functor on the category of finite CW-complexes and homotopy classes of (not necessarily cellular) maps. In a similar way, using the orthogonal projection $Z^i(X) \to \mathcal{H}_i(X)$, we obtain isomorphisms

$$H^i(X;\mathbb{R}) \xrightarrow{\cong} \mathcal{H}_i(X) \tag{3}$$

yielding **cofunctors** (i.e., contravariant functors) $\mathcal{H}_i$ with induced maps

$$f^!\colon \mathcal{H}_i(Y) \to \mathcal{H}_i(X),$$

and one checks easily that $f_!$ and $f^!$ are adjoints of each other.

1.5. To illustrate the use of harmonic chains, we consider the following example. Let $pr\colon \bar{X} \to X$ be the projection of a finite, regular, connected covering space, with X a finite CW-complex and $\bar{X}$ carrying the cell structure induced from the cell structure of the base space X. If $\bar{\sigma}$ denotes an i-cell over σ and $\bar{\tau}$ one over τ, the projection $pr_i\colon C_i\bar{X} \to C_iX$ satisfies

$$\langle pr_i\bar{\sigma}, \tau\rangle = \delta_{\sigma,\tau} = \langle \bar{\sigma}, \sum_{g\in G} g\bar{\tau}\rangle,$$

where G denotes the covering transformation group. It follows that the adjoint $(pr_i)^*\colon C_i(X) \to C_i(\bar{X})$ is given by $c \mapsto \sum_{g\in G} g\bar{c}$, where $pr_i\bar{c} = c$. We will use the notation $\sum g\bar{c} = N\bar{c}$, $N \in \mathbb{Z}[G]$ the norm element, thus

$$pr_i^* \circ pr_i = N, \quad pr_i \circ pr_i^* = |G|.$$

Therefore, the adjoint $pr_i^*\colon C_i(X) \to C_i(\bar{X})$ is injective, and

$$pr_i^*(pr_i \circ \delta_{i-1}) = N\delta_{i-1} = \delta_{i-1}N = pr_i^*(\delta_{i-1} \circ pr_i)$$

so that, in this case, pr_i commutes with δ and d, inducing

$$\mathcal{H}_i(pr_i) = pr_!\colon \mathcal{H}_i(\bar{X}) \to \mathcal{H}_i(X).$$

The adjoint $(pr_!)^* = pr_i^!$ is then induced by pr_i^* and satisfies $pr_! \circ pr^! = |G|$ as well as $pr^! \circ pr_! = N$. We thus obtain isomorphisms

$$pr_!\colon \mathcal{H}_i(\bar{X})^G \xrightarrow{\cong} \mathcal{H}_i(X), \quad \text{and} \quad pr^!\colon \mathcal{H}_i(X) \xrightarrow{\cong} \mathcal{H}_i(\bar{X})^G.$$

If one writes $\mathcal{H}_i(g)$ for the map $g_!\colon \mathcal{H}_i(\bar{X}) \to \mathcal{H}_i(\bar{X})$ induced by the covering transformation $g\colon \bar{X} \to \bar{X}, (g \in G)$, then $\mathcal{H}_i(\bar{X})$ is an $\mathbb{R}[G]$-module and

$$b_i(X) = \frac{1}{|G|}\sum_{g\in G} tr\big(\mathcal{H}_i(g)\big), \tag{4}$$

because $\frac{1}{|G|}\sum tr\big(\mathcal{H}_i(g)\big)$ equals the multiplicity of the trivial representation in the G-representation $\mathcal{H}_i(\bar{X})$.

Remark 1.5.1: The idea of using the natural inner product structure on chain groups goes back to [6], where the notion of harmonic chains for finite simplicial complexes was first introduced and discussed. Various applications, in particular of (4), can be found in [7].

2. Regular coverings of finite CW-complexes; ℓ_2-chains

Let Y be a connected CW-complex and G a group acting freely on Y by permuting the cells. We assume* the action on Y to be cocompact so that $X = Y/G$ is a finite CW-complex. Note that then G must be countable, being a factor group of the finitely generated fundamental group $\pi_1(X)$. We write $\ell_2 G$ for the (real separable) Hilbert space of square summable functions** $f\colon G \to \mathbb{R}$; sometimes we use the notation $\sum_{x\in G} f(x)x$ for such an f, with $f(x) \in \mathbb{R}$ and $\sum f(x)^2 < \infty$. (The general facts which follow do not depend on the condition of G being countable. If the discrete group G is not countable, $\ell_2 G$ is defined to be the Hilbert space of real valued functions on G with countable support.) The inner product on $\ell_2 G$ is given by

$$\ell_2 G \times \ell_2 G \xrightarrow{\langle\,,\,\rangle} \mathbb{R}, \quad \langle f, g\rangle = \sum_{x\in G} f(x)g(x).$$

Note that the group algebra $\mathbb{R}G$ can then be viewed as a dense subspace of $\ell_2 G$, consisting of all functions $G \to \mathbb{R}$ with *finite* support. In this way we may consider $G \subset \ell_2 G$ as a subset, and we write $1 \in \ell_2 G$ for the image of $1 \in G$. We like to stress that the inclusion $\mathbb{R}[G] \subset \ell_2 G$ is not an inclusion of rings: the multiplication in $\mathbb{R}[G]$ does in general not extend in a natural way to a multiplication in $\ell_2 G$. The elements $y \in G$ operate then via isometries on $\ell_2 G$, from the left and from the right: for $f \in \ell_2 G$ one has

$$y \cdot \sum_x f(x)x = \sum_x f(y^{-1}x)x, \quad \sum_x f(x)x \cdot y = \sum_x f(xy^{-1})x$$

yielding a $\mathbb{Z}[G]$-bimodule structure on $\ell_2 G$. Note also that the associated action of $\mathbb{R}G$ is an action by bounded operators. Indeed if $\alpha = \sum r(x)x \in \mathbb{R}G$ and $f \in \ell_2 G$, then

$$\|\alpha \cdot f\| \le \sum |r(x)| \|xf\| = |\alpha| \cdot \|f\|,$$

where $|\alpha| = (\sum |r(x)|)$; similarly for $\|f \cdot \alpha\|$.

2.2. Because G acts cocompactly on Y, the cellular chain group $K_i Y$ is a finitely generated free $\mathbb{Z}[G]$-module of rank equal to the number of i-cells of $X = Y/G$. We put

$$C_i(Y, G) = \ell_2 G \otimes_G K_i(Y).$$

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* Cf. Remark 2.6.3.

** For simplicity we work throughout over $\mathbb{R}$. Everything could be done over $\mathbb{C}$ (which is relevant in a more general context, but not here).

If the G-action on Y is clear from the context, we write just $C_i(Y)$ for $C_i(Y,G)$. Note that $C_i(Y)$ is a (left) $\mathbb{R}G$-module. We define a Hilbert space structure on $C_i(Y)$ by exhibiting an orthonormal Hilbert basis. For this we choose from each G-orbit of i-cells a representative $\bar{\tau}_i^\mu, \mu \in \{1,\dots,\alpha_i\}$, with α_i the number of i-cells of X. Then

$$\{x \otimes \bar{\tau}_i^\mu | x \in G, \mu \in \{1,\dots,\alpha_i\}\}$$

constitutes an orthonormal Hilbert basis for $C_i(Y)$; obviously, the Hilbert space structure on $C_i(Y)$ does not depend on the choice of the representatives $\bar{\tau}_i^\mu$. (As a matter of fact, $C_i(Y)$ is naturally isomorphic as a Hilbert space to the space of square summable chains

$$C_i^{(2)}(Y) := \{\sum_{\sigma\in J_i} f(\sigma)\sigma |\, f(\sigma) \in \mathbb{R}, \sum_{\sigma\in J_i} f(\sigma)^2 < \infty\},$$

with orthonormal Hilbert basis $\{\sigma\}_{\sigma\in J_i}$, J_i denoting the set of i-cells of Y.) Note also that for $f \in \ell_2 G$, the elements $f \otimes \bar{\tau}_i^\mu \in C_i(Y)$ satisfy $\|f \otimes \bar{\tau}_i^\mu\| = \|\sum f(x)x \otimes \bar{\tau}_i^\mu\| = \|f\|$ and therefore

$$\begin{aligned}(\ell_2 G)^{\alpha_i} &\to C_i(Y)\\ (f_1,\dots,f_{\alpha_i}) &\mapsto \sum_{\mu=1}^{\alpha_i} f_\mu \otimes \bar{\tau}_i^\mu\end{aligned}$$

defines an isometric G-equivariant isomorphism of Hilbert spaces (here $(\ell_2 G)^{\alpha_i}$ is considered as a Hilbert-space in the usual way, with

$$\|(f_1,\dots,f_{\alpha_i})\|^2 = \sum_{\mu=1}^{\alpha_i} \|f_\mu\|^2$$

so that the inclusions $\ell_2 G \to (\ell_2 G)^{\alpha_i}$ are isometric G-equivariant embeddings). The induced boundary maps

$$\ell_2 G \otimes_G d_i \colon C_i(Y) \to C_{i-1}(Y),$$

which we denote by d_i too, if no confusion can arise, are bounded operators. Indeed, the following more general result is easy to prove.

Lemma 2.2.1: *Let $\varphi\colon (\mathbb{R}G)^n \to (\mathbb{R}G)^m$ be a morphism of $\mathbb{R}G$-modules. Then the induced operator $\tilde{\varphi} := \ell_2 G \otimes_{\mathbb{R}G} \varphi\colon (\ell_2 G)^n \to (\ell_2 G)^m$ is bounded.*

Proof: Let $[\varphi_{ij}]$ denote the matrix of φ so that

$$\varphi(a_1,\dots,a_n) = (\sum a_i\varphi_{i1},\dots,\sum a_i\varphi_{im}), \quad (a_1,\dots,a_n) \in (\mathbb{R}G)^n$$

with $\varphi_{ij} \in \mathbb{R}G$. Each φ_{ij} has the form $\sum_x c_{ij}(x)x$, with $c_{ij}(x) \in \mathbb{R}$ and $x \in G$. We write as before

$$|\varphi_{ij}| = \sum_x |c_{ij}(x)|.$$

Then for $f \in \ell_2 G$ one has $\|f \cdot \varphi_{ij}\| \leq |\varphi_{ij}| \cdot \|f\|$, and

$$\begin{aligned} \|\tilde{\varphi}(f_1, \ldots, f_n)\|^2 &= \sum_j \| \sum_i f_i \varphi_{ij} \|^2 \\ &\leq \sum_{i,j} |\varphi_{ij}|^2 \|f_i\|^2 \\ &\leq (\sum_{i,j} |\varphi_{ij}|^2) \|(f_1, \ldots, f_n)\|^2. \end{aligned}$$

Thus $\tilde{\varphi}$ is bounded. ∎

2.3. In particular, the operators $d_i\colon C_i(Y) \to C_{i-1}(Y)$ and their adjoints $\delta_{i-1} = d_i^*$ are continuous. We put $\ker d_i = Z_i(Y)$, and $\ker \delta_i = Z^i(Y)$; these are thus closed subspaces of $C_i(Y)$ so that

$$\mathcal{H}_i(Y, G) := Z_i(Y) \cap Z^i(Y)$$

is a Hilbert subspace (i.e., a closed linear G-subspace) of $C_i(Y)$. The images $B_i(Y) := \mathrm{im}(d_{i+1}\colon C_{i+1}(Y) \to C_i(Y))$ and $B^i(Y) = \mathrm{im}(\delta_{i-1}\colon C_{i-1}(Y) \to C_i(Y))$ need not be closed; we write $\bar{B}_i$ and $\bar{B}^i$ for their closures, respectively. One finds then, as in 1.2, orthogonal decompositions

$$C_i(Y) = \bar{B}^i \perp Z_i = \bar{B}_i \perp Z^i = \bar{B}^i \perp \bar{B}_i \perp \mathcal{H}_i$$

(ℓ_2-Hodge–de Rham decomposition). As in the first section, we define the Laplacian by

$$\Delta_i = d_{i+1}\delta_i + \delta_{i-1}d_i\colon C_i(Y) \to C_i(Y)$$

and

$$\mathcal{H}_i(Y, G) = \{c \in C_i(Y) | \Delta_i c = 0\}.$$

The proof is essentially the same as in the situation described in section 1. One refers to $\mathcal{H}_i(Y, G)$ as the space of **harmonic** ℓ_2-chains on Y, and we will often just write $\mathcal{H}_i(Y)$, if the G-action is plain from the context.

By analogy with 1.3 it is now natural to introduce **reduced** ℓ_2-homology groups of Y by

$${}^{red}H_i(Y) = Z_i(Y)/\bar{B}_i(Y).$$

Then the projection $Z_i \to {}^{red}H_i(Y)$ induces an isomorphism $\mathcal{H}_i(Y) \cong {}^{red}H_i(Y)$ as Hilbert spaces. Similarly reduced cohomology groups are defined by

$$ {}^{red}H^i(Y) = Z^i(Y)/\bar{B}^i(Y) $$

and one has an isomorphism $\mathcal{H}_i(Y) \cong {}^{red}H^i(Y)$. In case of a finite group G it is clear from section 1 that

$$ \mathcal{H}_i(Y) \cong H_i(Y;\mathbb{R}) \cong H^i(Y;\mathbb{R}). $$

2.4. One might also look at non-reduced (co)-homology groups Z_i/B_i and Z^i/B^i. As for the homology groups,

$$ Z_i/B_i \cong H_i(\ell_2 G \otimes_G K_*(Y)) =: H_i^G(Y;\ell_2 G), $$

the **equivariant** homology of Y with coefficients the G-module $\ell_2 G$. There is thus a natural surjection

$$ H_i^G(Y;\ell_2 G) \to {}^{red}H_i(Y). $$

In a similar way, using the cochain complex

$$ C^i(Y) := \mathrm{Hom}_G(K_i(Y), \ell_2 G), $$

one defines equivariant cohomology groups

$$ H_G^i(Y;\ell_2 G) := H^i(C^*(Y)) = H^i(\mathrm{Hom}_G(K_*(Y), \ell_2 G)). $$

Because $K_i(Y)$ is finitely generated and free as a $\mathbb{Z}[G]$-module, $K_i(Y)$ is isomorphic to its dual $\mathrm{Hom}_G(K_i(Y), \mathbb{Z}[G])$. Similarly, using the Hilbert basis of $C_i(Y)$ corresponding to the i-cells of Y, we can identify the Hilbert spaces $C_i(Y)$ and $C^i(Y) = \mathrm{Hom}_G(K_i(Y), \ell_2 G)$. This leads to isomorphisms

$$ Z^i/B^i \cong H^i(\mathrm{Hom}_G(K_i(Y), \ell_2 G)) =: H_G^i(Y;\ell_2 G), $$

and a natural surjection

$$ H_G^i(Y;\ell_2 G) \to {}^{red}H^i(Y). $$

We also point out that, analogous to the isomorphism $C_i^{(2)}(Y) \cong C_i(Y)$, the cochains $C^i(Y)$ can be identified with the ℓ_2-cochains $C^i_{(2)}(Y)$ (defined as those real cochains ϕ: $K_i(Y) \to \mathbb{R}$, which are square summable, $\sum |\phi(\sigma)|^2 < \infty$, the sum being taken over all i-cells of Y). Clearly, $C^i_{(2)}(Y)$ is naturally isomorphic to the Hilbert space dual $\mathrm{Hom}_{cont}(C_i^{(2)}(Y),\mathbb{R})$, and one has isomorphisms of Hilbert spaces

$$^{red}H_i(Y) \cong \mathrm{Hom}_{cont}(^{red}H_i(Y),\mathbb{R}) \cong {}^{red}H^i(Y), \quad h \mapsto \langle h, \ \rangle.$$

Finally, we observe that there are obvious maps

$$\mathrm{can}_i : H_i(Y;\mathbb{R}) \longrightarrow {}^{red}H_i(Y)$$

given by considering an ordinary real cycle as an ℓ_2-cycle, and

$$\mathrm{can}^i : {}^{red}H^i(Y) \longrightarrow H^i(Y;\mathbb{R}),$$

which can be described as follows: $\mathrm{can}^i(x) = [\tilde{x}]$, where $\tilde{x}$ denotes the unique harmonic cocycle in $C^i_{(2)}$ representing x, and $[\tilde{x}]$ is its ordinary $\mathbb{R}$-cohomology class.

As is clear from examples (see 2.7.3), the unreduced groups are indeed different from the reduced ones, and they do not easily yield numerical invariants. The advantage of the reduced groups is that they are *Hilbert G-modules*, as explained in the next section.

2.5. If M is a Hilbert space, we call $V \subset M$ a **Hilbert subspace**, if V is a closed linear subspace with induced Hilbert space structure.

Definition 2.5.1: A Hilbert G-module is a left G-module M, which is a Hilbert space on which G acts by isometries such that M is isometrically G-isomorphic to a G-stable Hilbert subspace of $(\ell_2 G)^n$ for some n.

It follows that $C_i(Y), Z_i(Y), Z^i(Y), \bar{B}_i(Y), \bar{B}^i(Y)$ and $\mathcal{H}_i(Y)$ are all Hilbert G-modules. If M is a Hilbert G-module and $V \subset M$ a G-stable linear subspace, then $M/\bar{V}$, $\bar{V}$ the closure of V, has a natural Hilbert G-module structure, with norm given by

$$\|w\| = \min\{\|\tilde{w}\| \mid \pi(\tilde{w}) = w\},$$

where $\pi\colon M \to M/\bar{V}$ denotes the projection. Note that π induces a G-equivariant isometric isomorphism of Hilbert G-modules

$$V^{\perp} \xrightarrow{\cong} M/\bar{V}.$$

Definition 2.5.2: A map $f\colon M_1 \to M_2$ of Hilbert-G-modules is a

- weak isomorphism, if f is an injective, bounded G-equivariant operator, with $\mathrm{im}(f)$ dense in M_2;

- strong isomorphism, if f is an isometric G-equivariant isomorphism of Hilbert spaces.

Using the polar decomposition of bounded operators, one easily deduces the following crucial fact.

LEMMA 2.5.3: *Suppose there exists a weak isomorphism $M_1 \to M_2$ of Hilbert G-modules. Then there exists also a strong isomorphism.*

Proof: Let $f\colon M_1 \to M_2$ be a weak isomorphism. Then $\langle f^*fv, v\rangle$ is > 0 for all $v \in M_1 \setminus \{0\}$ and $f^*f\colon M_1 \to M_1$ is a positive operator with $\mathrm{im}(f^*f)$ dense. It follows that there exists a unique positive self-adjoint operator $g\colon M_1 \to M_1$ with $g^2 = f^*f$, and $\mathrm{im}(g) \supset \mathrm{im}(g^2)$ is dense in M_1. Put $\bar{h} = f \circ g^{-1}\colon \mathrm{im}(g) \to M_2$ (g^{-1} exists since g is injective). Then $\mathrm{im}(\bar{h}) = \mathrm{im}(f) \subset M_2$ is dense, and for $x, y \in \mathrm{im}(g)$

$$\begin{aligned}\langle \bar{h}x, \bar{h}y\rangle &= \langle f^*f \circ g^{-1}x, g^{-1}y\rangle \\ &= \langle g^2 \circ g^{-1}x, g^{-1}y\rangle = \langle g \circ g^{-1}x, g^* \circ g^{-1}y\rangle \\ &= \langle x, y\rangle,\end{aligned}$$

where we used the fact that $g^* = g$. It follows that $\bar{h}$ is an isometric isomorphism $\mathrm{im}(g) \to \mathrm{im}(f)$. Since $\mathrm{im}(g)$ is dense in M_1 and $\mathrm{im}(f)$ is dense in M_2, $\bar{h}$ extends by continuity to an isometric isomorphism $h\colon M_1 \to M_2$. Since f and f^* are G-equivariant, g is G-equivariant too and so is $\bar{h}$. It follows that h is a strong isomorphism of Hilbert G-modules. ∎

Definition 2.5.4: Two Hilbert G-modules M_1 and M_2 will be called isomorphic, and we will write $M_1 \cong M_2$, if there exists a weak and therefore also a strong isomorphism $M_1 \to M_2$.

COROLLARY 2.5.5: *Let $\varphi\colon M_1 \to M_2$ be a bounded G-equivariant operator of Hilbert G-modules. Then*

$$(\ker\varphi)^\perp \cong M_1/\ker\varphi \cong \overline{\mathrm{im}\,\varphi}$$

as Hilbert G-modules.

Proof: We have already seen that the projection $M_1 \to M_1/\ker\varphi$ defines a strong isomorphism $(\ker\varphi)^\perp \to M_1/\ker\varphi$. The canonical map $M_1/\ker\varphi \to \overline{\mathrm{im}\,\varphi}$ is a weak isomorphism. Thus $M_1/\ker\varphi \cong \overline{\mathrm{im}\,\varphi}$. ∎

2.6. Our next goal is to show that the isomorphism type of the Hilbert G-module $\mathcal{H}_i(Y)$ depends only upon the G-homotopy type of Y. It is plain that the projection $Z_i(Y) \to {}^{red}H_i(Y)$ induces a strong isomorphism of Hilbert G-modules

$$\mathcal{H}_i(Y) \xrightarrow{\cong} {}^{red}H_i(Y).$$

LEMMA 2.6.1: *${}^{red}H_i(\ )$ defines a functor from the category of free cocompact G-CW-complexes and G-homotopy classes of maps, to the category of Hilbert G-modules and bounded G-equivariant operators.*

In particular, the Hilbert G-modules $\mathcal{H}_i(Y)$ of harmonic ℓ_2-chains give rise to G-homotopy invariants.

COROLLARY 2.6.2: *If $f\colon Y \to Z$ is a G-map between free cocompact G-CW-complexes and f is a homotopy equivalence, then the Hilbert G-modules $\mathcal{H}_i(Y)$ and $\mathcal{H}_i(Z)$ are isomorphic.*

Indeed, f induces a weak equivalence ${}^{red}H_i(Y) \to {}^{red}H_i(Z)$ and therefore ${}^{red}H_i(Y) \cong {}^{red}H_i(Z)$, thus $\mathcal{H}_i(Y) \cong \mathcal{H}_i(Z)$ as Hilbert G-modules. (There is no need to assume that the map f in the corollary is a G-homotopy equivalence; as a matter of fact, it is well known that any G-map between free G-CW-complexes, which is a homotopy equivalence, is also a G-homotopy equivalence.)

Proof of the Lemma: Let $f\colon Y \to Z$ be a G-map of free cocompact G-CW-complexes. Then, by the G-cellular approximation theorem, f is G-homotopic to a cellular G-map $\tilde{f}\colon Y \to Z$, inducing bounded operators (cf. 2.2.1) and chain maps

$$\tilde{f}_i\colon C_i(Y) \to C_i(Z).$$

Since $\tilde{f}_i$ is continuous, it maps $\bar{B}_i(Y)$ to $\bar{B}_i(Z)$ and induces therefore ${}^{red}H_i(\tilde{f})\colon {}^{red}H_i(Y) \to {}^{red}H_i(Z)$. If $\tilde{\tilde{f}}\colon Y \to Z$ is a cellular G-map G-homotopic to $\tilde{f}$, then

$$K_i\tilde{f}, K_i\tilde{\tilde{f}}\colon K_iY \to K_iZ$$

are chain homotopic morphisms of G-chain complexes. It follows that $\ell_2G \otimes_{\mathbb{R}G} K_*\tilde{f} =: \tilde{f}_*$ and $\ell_2G \otimes_{\mathbb{R}G} K_*\tilde{\tilde{f}} =: \tilde{\tilde{f}}_*$ are chain homotopic too. Thus $(\tilde{f}_* - \tilde{\tilde{f}}_*)(Z_i(Y)) \subset B_i(Y) \subset \bar{B}_i(Y)$ and, therefore, ${}^{red}H_i(\tilde{f}_*) = {}^{red}H_i(\tilde{\tilde{f}}_*)$ for all i, showing that ${}^{red}H_i(\tilde{f}_*)$ depends on the G-homotopy class of f only. ∎

Similarly the reduced ℓ_2-cohomology groups,

$${}^{red}H^i(Y) = Z^i(Y)/\bar{B}^i(Y),$$

define cofunctors from free cocompact G-CW-complexes (and G-homotopy classes of G-maps) to the category of Hilbert G-modules (and bounded G-equivariant operators), with $f\colon Y \to Z$ inducing

$$^{red}H^i(f)\colon {}^{red}H^i(Z) \to {}^{red}H^i(Y)$$

via the adjoint $f_i^*\colon C_i(Z) \to C_i(Y)$. The reader should be careful to notice that the left G-module structure on $^{red}H^i(Y)$, which we are considering, is the one induced from considering the Hilbert G-module $^{red}H^i(Y)$ as a G-subspace of $C_i(Y)$. We could also let $g \in G$ act via $^{red}H^i(g)$, which would define a **right** G-action on $^{red}H^i(Y)$. Passing to the associated left G-action using the inverse, would result in the left G-action considered first, because G acts by isometries on $^{red}H^i(Y)$ and therefore the adjoint of $^{red}H^i(g)$ equals its inverse. In any case, the left G-spaces $^{red}H_i(Y)$, $\mathcal{H}_i(Y)$, and $^{red}H^i(Y)$ are all isomorphic as Hilbert G-modules.

Remark 2.6.3: One can extend the definitions of the previous sections in the following obvious way. If Y is a G-CW-complex such that for a given $k \in \mathbb{N}$ the k-skeleton Y^k is a free cocompact G-CW-complex, then we define for $i < k$ the i-th (co)-homology groups (with all variations considered above), to be those of Y^k. In particular, for any finitely presented group G, the groups $\mathcal{H}_0(EG)$ and $\mathcal{H}_1(EG)$ are well defined, by choosing a model for EG with cocompact 2-skeleton.

2.7. The following examples serve as an illustration. But first we need an elementary fact.

LEMMA 2.7.1: *If G is an infinite group, then for $n \geq 1$ the left G-module $(\ell_2 G)^n$ contains no G-invariant element $\neq 0$.*

Proof: If $\sum_{x\in G} f(x)x \in \ell_2 G$ is G-invariant, then $f(x)$ must be independent of x and, since G is infinite, $f(x)$ must be zero for all x; similarly for the case $n > 1$. ∎

Example 2.7.2: Let Y be a connected G-CW-complex with cocompact 1-skeleton and G an infinite group. Then $^{red}H_0(Y) = \mathcal{H}_0(Y) = {}^{red}H^0(Y) = 0$. To see this, we consider the right-exact sequence

$$C_1(Y) \xrightarrow{d_1} C_0(Y) \longrightarrow \ell_2 G \otimes_G \mathbb{Z} \longrightarrow 0.$$

It shows that $\ker \delta_0 = (\operatorname{im} d_1)^\perp$ gets mapped injectively into the coinvariants $\ell_2 G \otimes_G \mathbb{Z}$, showing that $\ker \delta_0$ consists of G-invariant elements and is therefore trivial. Whence, ${}^{red}H^0(Y) = 0$. In particular, for any finitely generated infinite group G one has $\mathcal{H}_0(EG) = 0$ (for G finite, $\mathcal{H}_0(EG) = \mathbb{R}$).

But the *unreduced* H_0 need not be zero, as the following example shows.

Example 2.7.3: Let $Y = \mathbb{R}$, the universal cover of S^1, where $G = \mathbb{Z}$ acts by covering transformations on $\mathbb{R}$ and Y/G is considered as a CW-complex with 2 cells: $S^1 = e^0 \cup e^1$. The complex of ℓ_2-chains $C_*(Y)$ then takes the form

$$0 \longrightarrow \ell_2(\mathbb{Z}) \xrightarrow{d_1} \ell_2(\mathbb{Z}) \longrightarrow 0.$$

If x denotes a generator of $\mathbb{Z}$ we can write $f \in \ell_2(\mathbb{Z})$ as $f = \sum_{n\in\mathbb{Z}} a_n x^n$, and d_1 is given by

$$d_1(f) = (1-x)\sum_{n\in\mathbb{Z}} a_n x^n \in \ell_2(\mathbb{Z}).$$

Clearly, d_1 is injective, and $\operatorname{im} d_1$ is dense in $\ell_2(\mathbb{Z})$ by 2.7.2. However, d_1 is not surjective. For instance, $1 \in \ell_2(\mathbb{Z})$ is not in the image of d_1, because $1 = (1-x)\sum_{i\in\mathbb{Z}} a_i x^i$ would imply that all a_j, $j < 0$, are equal whence 0, and all $a_j, j \geq 0$ are equal whence 0, which is not possible. Therefore ${}^{red}H_*(Y) = 0$ whereas

$$Z_0(Y)/B_0(Y) = H_0^G(Y;\ell_2 G) \neq 0.$$

Expressed in another way, the example shows that $\mathcal{H}_i(E\mathbb{Z}) = 0$ for all $i \geq 0$, whereas $H_i^{\mathbb{Z}}(E\mathbb{Z};\ell_2\mathbb{Z}) = H_i(\mathbb{Z};\ell_2\mathbb{Z}) = 0$ for $i > 0$, and $H_0(\mathbb{Z};\ell_2\mathbb{Z}) \neq 0$ (and $H^i(\mathbb{Z};\ell_2\mathbb{Z}) = 0$ for $i \neq 1$, $H^1(\mathbb{Z};\ell_2\mathbb{Z}) \neq 0$).

Remark 2.7.4: A systematic study of the difference between reduced and unreduced ℓ_2-homology leads to the notion of *torsion Hilbert modules*, with associated *Novikov–Shubin* invariants (cf. [21]). For a systematic treatment of these matters the reader is referred to [13, 19].

3. Von Neumann dimension; ℓ_2-Betti numbers

The goal of this section is to define a real valued function "$\dim_G$" (**von Neumann dimension**) on Hilbert G-modules satisfying the following basic properties:

- $\dim_G M \geq 0$,
- $\dim_G M = 0 \Longleftrightarrow M = 0$,
- $\dim_G M = \dim_G N$ if $M \cong N$,
- $\dim_G M \oplus N = \dim_G M + \dim_G N$,
- $\dim_G M \leq \dim_G N$ if $M \subset N$,

- $\dim_G \ell_2 G = 1$,
- $\dim_G M = \frac{1}{|G|} \dim_{\mathbb{R}} M$, if G is finite,
- $\dim_G M = \frac{1}{[G:S]} \dim_S M$, if $S < G$ has finite index.

The function $\dim_G$ will be derived from a generalization of the standard Kaplansky trace map

$$\rho\colon \mathbb{R}G \longrightarrow \mathbb{R}, \quad \sum_{x \in G} r(x)x \longmapsto r(1),$$

with $1 \in G$ the neutral element.

Definition 3.1.1: The von Neumann algebra $N(G)$ is the algebra of bounded (left) G-equivariant operators $\ell_2 G \to \ell_2 G$.

Recall that $\ell_2 G$ is an $\mathbb{R}G$-bimodule. Since the right action of $\mathbb{R}G$ on $\ell_2 G$ is an action by bounded left G-equivariant operators, we may consider $\mathbb{R}G$ as a subalgebra of the von Neumann algebra $N(G)$. Mapping an operator ϕ to its adjoint ϕ^* defines an involution on $N(G)$ (turning it into a real C^*-algebra). Because the adjoint of the right action of $x \in G$ on $\ell_2 G$ is right multiplication by x^{-1}, passing to the adjoint in $N(G)$ corresponds under the inclusion $\mathbb{R}G \subset N(G)$ to **conjugation** in $\mathbb{R}G$. By conjugation on $\mathbb{R}G$ (or $\ell_2 G$) we mean the map

$$f = \sum f(x)x \longmapsto \bar{f} = \sum f(x)x^{-1}.$$

The Kaplansky trace on $\mathbb{R}G$ can now be extended to a **trace** on $N(G)$ as follows.

Definition 3.1.2: Let $\varphi \in N(G)$ and $1 \in \mathbb{R}G \subset \ell_2 G$. Then

$$\operatorname{trace}_G(\varphi) = \langle \varphi(1), 1 \rangle \in \mathbb{R}.$$

It follows that if we consider $w = \Sigma r(x)x \in \mathbb{R}G$ as an element of $N(G)$,

$$\operatorname{trace}_G(w) = \langle 1 \cdot \Sigma r(x)x, 1 \rangle = \langle \Sigma r(x)x, 1 \rangle = r(1) = \rho(w)$$

where, as before, $\rho(w)$ stands for the Kaplansky trace of w. We also observe that the trace of $\varphi \in N(G)$ satisfies

$$\operatorname{trace}_G(\varphi) = \langle \varphi(1), 1 \rangle = \langle 1, \varphi^*(1) \rangle = \operatorname{trace}_G(\varphi^*).$$

The following lemma shows that the inclusion $\mathbb{R}G \subset \ell_2 G$ extends to an embedding of G-modules $N(G) \subset \ell_2 G$ ($\ell_2 G$ is not a ring!) under which the $*$-involution of $N(G)$ corresponds to conjugation in $\ell_2 G$; in the course of its proof we will make use of the obvious fact that for $f, g \in \ell_2 G$ conjugation satisfies $\langle f, g \rangle = \langle \bar{f}, \bar{g} \rangle$.

LEMMA 3.1.3: *The $\mathbb{R}$-linear map*

$$\theta\colon N(G) \to \ell_2(G), \quad \varphi \longmapsto \varphi(1),$$

is injective and satisfies $\theta(\varphi^*) = \overline{\varphi(1)}$.

Proof: Let $\varphi \in N(G)$ be such that $\varphi(1) = 0$. Then, for all $x \in G \subset \ell_2 G$ one has $\varphi(x) = x\varphi(1) = 0$, showing that θ is injective. Furthermore, for all $x \in G$

$$\begin{aligned}\langle \varphi^*(1), x\rangle =&\langle 1, \varphi(x)\rangle = \langle 1, x\varphi(1)\rangle \\ =&\langle x^{-1}, \varphi(1)\rangle = \langle \bar{x}, \varphi(1)\rangle = \langle \varphi(1), \bar{x}\rangle \\ =&\langle \overline{\varphi(1)}, x\rangle,\end{aligned}$$

and it follows that $\varphi^*(1) = \overline{\varphi(1)}$. ∎

Our "trace_G" has indeed the basic property one requires of a trace:

COROLLARY 3.1.4: *Let $\varphi, \psi \in N(G)$. Then*

$$\mathrm{trace}_G(\varphi\psi) = \mathrm{trace}_G(\psi\varphi).$$

Proof: We have

$$\begin{aligned}\mathrm{trace}_G(\varphi\psi) =&\langle \varphi(\psi(1)), 1\rangle = \langle \psi(1), \varphi^*(1)\rangle \\ =&\langle \psi(1), \overline{\varphi(1)}\rangle = \langle \overline{\psi(1)}, \varphi(1)\rangle \\ =&\langle \psi^*(1), \varphi(1)\rangle = \langle 1, \psi(\varphi(1))\rangle \\ =&\,\mathrm{trace}_G(\psi\varphi).\end{aligned}$$ ∎

3.2. Let $\mathrm{M}_n(N(G))$ denote the algebra of bounded (left) G-equivariant operators $(\ell_2 G)^n \to (\ell_2 G)^n$. An operator $F \in \mathrm{M}_n(N(G))$ is determined in the usual way by a matrix $[F_{i,j}]$ of operators $F_{i,j}$ in $N(G)$ satisfying

$$F(a_1, \ldots, a_n) = \left(\sum F_{1,k}a_k, \ldots, \sum F_{n,k}a_k\right) \in (\ell_2 G)^n.$$

Note that the adjoint F^* corresponds to the matrix of operators $[(F^*)_{i,j}] = [F^*_{j,i}]$. We extend the definition of the trace to operators F in $\mathrm{M}_n(N(G))$ by putting

$$\mathrm{trace}_G(F) := \sum_{i=1}^{n} \mathrm{trace}_G(F_{i,i}).$$

Clearly $\mathrm{trace}_G(F) = \mathrm{trace}_G(F^*)$ and, using Corollary 3.1.4, we see that for all $F_1, F_2 \in \mathrm{M}_n(N(G))$

$$\mathrm{trace}_G(F_1 \circ F_2) = \mathrm{trace}_G(F_2 \circ F_1).$$

LEMMA 3.2.1: *If $F \in \mathrm{M}_n(N(G))$ is self-adjoint and idempotent, then*

$$\operatorname{trace}_G(F) = \sum_{i,j} \|F_{i,j}(1)\|^2.$$

Proof: It follows from the definitions that

$$\begin{aligned}\operatorname{trace}_G(F) &= \sum_j \langle F_{j,j}(1), 1\rangle = \sum_j \langle (F^2)_{j,j}(1), 1\rangle \\ &= \sum_{i,j} \langle F_{j,i}F_{i,j}(1), 1\rangle = \sum_{i,j} \langle F_{i,j}(1), F^*_{j,i}(1)\rangle \\ &= \sum_{i,j} \langle F_{i,j}(1), F_{i,j}(1)\rangle = \sum_{i,j} \|F_{i,j}(1)\|^2. \qquad \blacksquare\end{aligned}$$

The following is a simple but important consequence.

COROLLARY 3.2.2: *Let $F \in \mathrm{M}_n(N(G))$ be self-adjoint and idempotent. Then* $\operatorname{trace}_G F$ *is non-negative and* $\operatorname{trace}_G F = 0$ *implies* $F = 0$.

Remark 3.2.3: The corollary holds too if F is only assumed to be an idempotent. This is easily seen by using the orthogonal projection $\pi \in \mathrm{M}_n(N(G))$ onto $\operatorname{im} F$, observing that $F \circ \pi = \pi$ and $\pi \circ F = F$ so that $\operatorname{trace}_G \pi = \operatorname{trace}_G F$. If $e \in N(G)$ is an idempotent, then $\operatorname{trace}_G(e) + \operatorname{trace}_G(1-e) = 1$ so that

$$\operatorname{trace}_G e \leq 1, \quad \text{and} \quad \operatorname{trace}_G e = 1 \Rightarrow e = 1.$$

Thus, if $e \in \mathbb{Z}[G]$ is an idempotent then, since in that case $\operatorname{trace}_G e$ must be an integer, one has $e = 0$ or $e = 1$, yielding the following classical result:

- the only idempotents in $\mathbb{Z}[G]$ are 0 and 1; or, equivalently, the integral group ring $\mathbb{Z}[G]$ does not admit a non-trivial decomposition into a direct sum of two left ideals.

The **Kaplansky Conjecture** states that the same conclusion holds for $\mathbb{R}[G]$ too, if one assumes G to be torsion free. This would follow, if one could prove that in the torsion free case $\operatorname{trace}_G e$ is an integer for any idempotent $e \in \mathbb{R}[G]$. It is known (Zalesskii [23], see also [2]) that for an arbitrary group G and idempotent $e \in \mathbb{R}[G]$ the value $\operatorname{trace}_G e$ is a rational number.

3.3. We are now ready to define the function $\dim_G$. First we consider a special case. Let $V \subset (\ell_2 G)^n$ be a G-invariant Hilbert subspace and let π_V denote the orthogonal projection onto V. Because for all $a \in (\ell_2)^n$ and $x \in G$

$$xa = \pi_V(xa) + (xa - \pi_V(xa)) = x\pi_V(a) + (xa - x\pi_V(a))$$

with $\pi_V(xa), x\pi_V(a) \in V$ and $xa - \pi_V(xa), xa - x\pi_V(a) \in V^\perp$, it follows that $\pi_V(xa) = x\pi_V(a)$ and therefore $\pi_V \in \mathrm{M}_n(N(G))$. The von Neumann dimension of V is now defined by

$$\dim_G V := \operatorname{trace}_G \pi_V \in \mathbb{R}.$$

Because π_V is a self-adjoint idempotent, we infer from Corollary 3.2.2 that $\dim_G V \geq 0$ and that $\dim_G V = 0$ implies $V = 0$.

For the general case we proceed as follows. Let M be an arbitrary Hilbert G-module and choose a G-equivariant isometric isomorphism

$$\alpha\colon M \xrightarrow{\cong} V \subset (\ell_2 G)^n.$$

Define the von Neumann dimension of M by

$$\dim_G M := \dim_G V.$$

We need to check that $\dim_G M$ does not depend on the choice of α. Suppose

$$\beta\colon M \xrightarrow{\cong} W \subset (\ell_2 G)^m$$

is another G-equivariant isometric isomorphism, with $m = n + k \geq n$. If V' denotes V considered as a subspace of $(\ell_2 G)^{n+k}$ using the inclusion

$$(\ell_2 G^n) \subset (\ell_2 G)^n \oplus (\ell_2 G)^k, \quad z \mapsto (z, 0),$$

one sees from the definition that $\dim_G V' = \dim_G V$. Therefore we may assume without loss of generality that $m = n$. Define then $h := \beta \circ \alpha^{-1}\colon V \to W$ and extend h to an operator $H \in \mathrm{M}_n(N(G))$ by putting $H|V = h$ and $H|V^\perp = 0$, so H is a *partial isometry* $V \to W$. The composition H^*H is, by construction, the orthogonal projection onto V, and HH^* is the orthogonal projection onto W. It follows that

$$\dim_G V = \operatorname{trace}_G(H^*H) = \operatorname{trace}_G(HH^*) = \dim_G W,$$

showing that $\dim_G M$ is well-defined indeed.

An immediate consequence of the definition is that

- $\dim_G M \geq 0$, and ($\dim_G M = 0 \iff M = 0$),
- $\dim_G M \oplus N = \dim_G M + \dim_G N$.

If $S < G$ has finite index m, then $G = \coprod_{i=1}^m Sx_i$, and $\ell_2 G$ decomposes as a Hilbert S-module into $\perp_{i=1}^m \ell_2 S \cdot x_i \cong (\ell_2 S)^m$. Thus if $F \in N(G)$ then

$$\operatorname{trace}_S F = \sum_{i=1}^m \langle F(x_i), x_i \rangle = \sum_{i=1}^m \langle F(1), 1 \rangle = m \cdot \operatorname{trace}_G F.$$

Similarly for $F \in M_n(N(G))$, so that for any Hilbert G-module, $\dim_S M = m \dim_G M$. The other properties stated in 3.1 are readily checked too and are left to the reader.

Remark 3.3.1: The general dimension theory in (complex, finite) von Neumann algebras goes back to the fundamental paper [20]. The dimension $\dim_G$ is closely related to the universal center-valued trace, and all our properties could be derived from it. The center-valued trace, however, is a rather difficult and deep concept while our treatment of $\dim_G$ is direct and elementary.

3.4. In applications, we will be dealing with chain complexes of Hilbert G-modules.

Definition 3.4.1: A chain complex

$$V_*\colon \cdots \to V_{i+1} \to V_i \to V_{i-1} \to \cdots$$

of Hilbert G-modules is called an **ℓ_2G-chain complex** if each $V_i \to V_{i-1}$ is a bounded G-equivariant operator.

Definition 3.4.2: Let

$$V_*\colon \cdots \to V_{i+1} \to V_i \to V_{i-1} \cdots$$

be an ℓ_2G-chain complex. Then

- the **reduced homology modules** of V_* are the Hilbert G-modules

$$^{red}H_i(V_*) = \ker(V_i \to V_{i-1})/\overline{\operatorname{im}(V_{i+1} \to V_i)};$$

- the complex V_* is called **weak-exact**, if $^{red}H_i(V_*) = 0$ for all i.

Definition 3.4.3: Let V_* and W_* be two ℓ_2G-chain complexes.

- A **morphism** $\phi_*\colon V_* \to W_*$ is an ordinary morphism of chain complexes consisting of bounded G-equivariant operators.
- Two morphisms $\phi_*, \psi_*\colon V_* \to W_*$ are ℓ_2G-homotopic if they are chain homotopic by a chain homotopy consisting of bounded G-equivariant operators.
- The complexes V_* and W_* are ℓ_2G-homotopy equivalent if there are morphisms $\phi_*\colon V_* \to W_*$ and $\psi_*\colon W_* \to V_*$ such that $\phi_* \circ \psi_*$ and $\psi_* \circ \phi_*$ are ℓ_2G-homotopic to the identity.

Clearly, a morphism $\phi_*\colon V_* \to W_*$ induces bounded G-equivariant operators $^{red}H_i(V_*) \to {}^{red}H_i(W_*)$, depending on the ℓ_2G-homotopy class of ϕ_* only. Indeed, the components of ϕ_* map cycles to cycles and, because of continuity, the closure of boundaries to the closure of boundaries; if ϕ_* is ℓ_2G-homotopic to $\tilde{\phi}_*$ then $\phi_i - \tilde{\phi}_i$ maps cycles to boundaries and thus they agree on reduced cohomology.

COROLLARY 3.4.4: *If the ℓ_2G-chain complexes V_* and W_* are ℓ_2G-homotopic, then the Hilbert G-modules ${}^{red}H_i(V_*)$ and ${}^{red}H_i(W_*)$ are isomorphic for all i.*

Definiton 3.4.5: A sequence $U \to V \to W$ of Hilbert G-modules is called **short weak-exact**, if

$$0 \to U \to V \to W \to 0$$

is a weak-exact ℓ_2G-chain complex.

Recall that for a G-equivariant bounded operator $\alpha : V \to W$ of Hilbert G-modules, the Hilbert G-modules $\overline{\alpha(V)}$ and $(\ker\alpha)^{\perp} \subset V$ are isomorphic, and the latter is isomorphic to $V/\ker\alpha$. It follows that

$$\dim_G V = \dim_G(\ker\alpha) + \dim_G(\overline{\alpha(V)}) = \dim_G(\ker\alpha) + \dim_G(V/\ker\alpha).$$

In particular, if $U \to V \to W$ is a short weak-exact sequence of Hilbert G-modules, then

$$\dim_G V = \dim_G U + \dim_G W.$$

COROLLARY 3.4.6: *Let*

$$V_*\colon 0 \to V_n \to V_{n-1} \to \cdots \to V_0 \to 0$$

be a chain complex of Hilbert G-Modules. Then

$$\sum_i (-1)^i \dim_G V_i = \sum_i (-1)^i \dim_G {}^{red}H_i(V_*).$$

Proof: Let $K_i = \ker(V_i \to V_{i-1})$ and $I_i = \overline{\operatorname{im}(V_{i+1} \to V_i)}$. Then there are short weak-exact sequences of Hilbert G-modules

$$K_i \to V_i \to I_{i-1}, \quad I_i \to K_i \to {}^{red}H_i(V_*),$$

yielding the equations

$$\dim_G V_i = \dim_G K_i + \dim_G I_{i-1}, \quad \dim_G {}^{red}H_i = \dim_G K_i - \dim_G I_i.$$

The result now follows readily. ∎

3.5. Let Y be a free cocompact G-CW-complex with cellular chain complex $K_*(Y)$. Then $C_*(Y) = \ell_2 G \otimes_G K_*(Y)$ is an $\ell_2 G$-chain complex in the sense of Definition 3.4.1. The i-th **ℓ_2-Betti number** of Y (with respect to G) is defined by

$$\beta_i(Y;G) := \dim_G {}^{red}H_i(Y).$$

It is plain from the results proved earlier that the ℓ_2-Betti numbers enjoy the following properties:

- $\beta_i(Y;G)$ is a G-homotopy invariant of Y (and therefore a homotopy invariant of the orbit space Y/G; cf. 3.4.4);
- if $S < G$ is a subgroup of index m, then $\beta_i(Y;S) = m \cdot \beta_i(Y;G)$ (in this case Y/S is an m-sheeted finite covering space of Y/G);
- if G is finite, then $\beta_i(Y;G) = \frac{1}{|G|} b_i(Y)$, where $b_i(Y)$ stands for the ordinary ith Betti number of Y (in this case, $\dim_G {}^{red}H_i(Y) = \frac{1}{|G|} \dim_{\mathbb{R}} H_i(Y;\mathbb{R})$); in particular if Y is connected, $\beta_0(Y;G) = \frac{1}{|G|}$;
- if G is infinite and Y is connected, then $\beta_0(Y;G) = 0$ (cf. 2.7.2).

Definition 3.5.1: Let X be a connected finite CW-complex. The **ℓ_2-Betti number** $\beta_i(X)$ of X is $\beta_i(\tilde{X};G)$, where $\tilde{X}$ denotes the universal covering space of X and $G = \pi_1(X)$.

Note that if α_i denotes the number of i-cells of X, the Hilbert G-module ${}^{red}H_i(\tilde{X})$ is isomorphic to a Hilbert submodule of $C_i(\tilde{X}) \cong (\ell_2 G)^{\alpha_i}$. Therefore the ℓ_2-Betti numbers satisfy

$$0 \leq \beta_i(X) \leq \alpha_i.$$

Furthermore, if $\bar{X}$ is a connected m-sheeted covering space of the finite complex X and $S < G$ denotes the fundamental group of $\bar{X}$, then

$$\beta_i(\bar{X}) = \dim_S {}^{red}H_i(\tilde{X}) = m \cdot \dim_G {}^{red}H_i(\tilde{X}) = m \cdot \beta_i(X),$$

which is quite different from the way the ordinary Betti numbers behave.

Example 3.5.2: If $X = S^1$, then $\beta_i(X) = 0$ for all $i \geq 0$ (cf. 2.7.3). More generally, if X is a connected finite CW-complex which possesses a regular finite covering space $\bar{X} \to X$ of degree $m > 1$ with $\bar{X}$ homotopy equivalent to X, then the ℓ_2-Betti numbers of X all vanish, because in this case $\beta_i(X) = \beta_i(\bar{X}) = m \cdot \beta_i(X)$.

3.6. Let X be a finite connected CW-complex with ordinary Betti numbers $b_i(X)$ and Euler characteristic $\chi(X) = \sum_i(-1)^i\alpha_i = \sum_i(-1)^i b_i(X)$, α_i the number of i-cells of X. It is an interesting fact that $\chi(X)$ can also be computed using the ℓ_2-Betti numbers.

THEOREM 3.6.1: *The Euler characteristic $\chi(X)$ of a finite connected CW-complex satisfies*

$$\chi(X) = \sum_i (-1)^i \beta_i(X).$$

Proof: We consider the ℓ_2-chain complex $C_*(\tilde{X}) = \ell_2 G \otimes_G K_*(\tilde{X})$, where G denotes the fundamental group of X. Since $C_i(\tilde{X}) \cong (\ell_2 G)^{\alpha_i}$ with α_i the number of i-cells of X,

$$\sum_i (-1)^i \dim_G C_i(\tilde{X}) = \sum_i (-1)^i \alpha_i = \chi(X).$$

On the other hand (cf. 3.4.6)

$$\sum_i (-1)^i \dim_G C_i(\tilde{X}) = \sum_i (-1)^i \dim_G {}^{red}H_i(\tilde{X}),$$

which proves the claim. ∎

The following is a slight generalization.

THEOREM 3.6.2: *Let X be a finite connected CW-complex and N a normal subgroup of $\pi_1 X$ with quotient group Q. Let X_N denote the covering space of X associated with N. Then*

$$\chi(X) = \sum (-1)^i \beta_i(X_N; Q).$$

Along the lines of 3.6.1 one can also establish the following **Morse Inequalities** (cf. 1.2.2).

COROLLARY 3.6.3: *Let X be a connected CW-complex with finite $(k+1)$-skeleton. Denote by α_i the number of i-cells and by β_i the ℓ_2-Betti numbers of X. Then*

$$\alpha_k - \alpha_{k-1} + \cdots + (-1)^k \alpha_0 \geq \beta_k - \beta_{k-1} + \cdots + (-1)^k \beta_0.$$

Remark 3.6.4: We also like to mention (without proof) the following **Künneth Formula** for ℓ_2-Betti numbers. Let X be a free cocompact G-CW-complex, and Y a free cocompact H-CW-complex. Then, using the fact that $X \times Y$ is a free cocompact $(G \times H)$-CW-complex with $K_*(X \times Y)$ and $K(X) \otimes_{\mathbb{Z}} K_*(Y)$ isomorphic as $G \times H$-complexes, one can show that

$$\beta_j(X \times Y; G \times H) = \sum_{s+t=j} \beta_s(X;G) \cdot \beta_t(Y;H).$$

(*Question*: If $F \to E \to B$ is a fibration, under what condition is $\beta_j E = \sum_{s+t=j} \beta_s F \cdot \beta_t B$? For the case of $\beta_1 E$, see [18].)

3.7. Let X be a connected CW-complex with fundamental group $\pi_1(X) =: G$. As earlier, we denote by $K_*(\tilde{X})$ the cellular chain complex of the universal cover $\tilde{X}$ of X. The associated n-**dual** ${}^nDK_*(\tilde{X})$ is defined by

$${}^nDK_j(\tilde{X}) = \mathrm{Hom}_G(K_{n-j}(\tilde{X}), \mathbb{Z}[G])$$

which we consider as (left) G-modules via $(xf)(c) := f(c)x^{-1}$, where $x \in G$ and $f \in \mathrm{Hom}_G(K_{n-j}(\tilde{X}), \mathbb{Z}[G])$; the differential is the one induced from $K_*(\tilde{X})$.

Definiton 3.7.1: We call a connected CW-complex X a **virtual** PD^n-complex, if there is a subgroup S of finite index in $\pi_1(X)$ such that $K_*(\tilde{X})$ is chain homotopy equivalent as a $\mathbb{Z}[S]$-complex to its n-dual ${}^nDK_*(\tilde{X})$. A group G is called a **virtual** PD^n-group, if $K(G,1)$ is a virtual PD^n-complex.

For instance, a closed (not necessarily orientable) topological n-manifold is homotopy equivalent to a virtual PD^n-complex in the above sense. Also, the reader checks easily that a group G of type FP_∞ is a virtual PD^n-group in the sense of (3.7.1) if and only if it possesses a subgroup of finite index which is a PD^n-group in the "usual" sense (as defined for instance in [12]).

THEOREM 3.7.2: *Let X be a finite virtual PD^n-complex. Then there exists a subgroup of finite index $S < \pi_1(X)$ such that the Hilbert S-modules ${}^{red}H_i(\tilde{X})$ and ${}^{red}H_{n-i}(\tilde{X})$ are isomorphic; in particular $\beta_i(X) = \beta_{n-i}(X)$ for all i and if $\pi_1(X)$ is infinite $\beta_n(X) = \beta_0(X) = 0$.*

Proof: Let $S < G$ be a subgroup of finite index such that the $\mathbb{Z}[S]$-complexes $K_*(\tilde{X})$ and ${}^nDK_*(\tilde{X})$ are chain homotopic. As in Section 2 we put $C_*(\tilde{X}) = \ell_2 S \otimes_S K_*(\tilde{X})$; define furthermore the $\ell_2 S$-chain complex ${}^nDC_*(\tilde{X})$ by

$${}^nDC_j(\tilde{X}) = \mathrm{Hom}_S(K_{n-j}(\tilde{X}), \ell_2 S),$$

with S-action on $f \in \mathrm{Hom}_S(K_{n-j}(\tilde{X}, \ell_2 S)$ given by $(xf)(c) = f(c)x^{-1}$ for $x \in S$ and $c \in K_{n-j}(\tilde{X})$, and obvious differential. Since $K_*(\tilde{X})$ is chain homotopy equivalent to ${}^nDK_*(\tilde{X})$ as $\mathbb{Z}[S]$-complex, the $\ell_2 S$-chain complexes $C_*(\tilde{X})$ and

${}^nDC_*(\tilde{X})$ are $\ell_2 S$-homotopic (cf. (3.4.3)). Thus by (2.6.2), the Hilbert S-modules ${}^{red}H_i(\tilde{X})$ and ${}^{red}H_i({}^nDC_*(\tilde{X}))$ are isomorphic. But

$$ {}^{red}H_i({}^nDC_*(\tilde{X})) \cong {}^{red}H^{n-i}(\tilde{X}) \cong {}^{red}H_{n-i}(\tilde{X}), $$

with the first isomorphism following from the definition of the n-dual complex, and the second one was discussed at the end of (2.6). Since

$$ \beta_j(X) = \dim_G {}^{red}H_j(\tilde{X}) = \frac{1}{[G:S]} \dim_S {}^{red}H_j(\tilde{X}), $$

the assertion concerning the ℓ_2-Betti numbers follows. ∎

3.8. If G is a group with a finite CW-model $K(G,1)$, we define its ℓ_2-Betti numbers by

$$ \beta_i(G) = \beta_i(K(G,1)). $$

According to 2.6.3 we can extend this definition as follows. Suppose G has a CW-model with finite n-skeleton for some $n \geq 2$ (i.e., G is of type F_n). Then we put

$$ \beta_i(G) = \beta_i(K(G,1)^n), \quad i < n. $$

In particular, $\beta_1(G)$ is defined for any finitely presented group G.

Example 3.8.1: Let $X = \vee_k S^1$ be a wedge of k circles. Then $\chi(X) = k-1$ and $X = K(*_k\mathbb{Z}, 1)$ so that

$$ \beta_i(\vee_k S^1) = \beta_i(*_k\mathbb{Z}) = \begin{cases} k-1, & \text{for } i = 1, \\ 0, & \text{else.} \end{cases} $$

In particular, $\beta_i(S^1) = \beta_i(\mathbb{Z}) = 0$ for all i. Using the Künneth formula we conclude that for any group G of type F_n

$$ \beta_i(\mathbb{Z} \times G) = 0 \quad \text{for } i < n. $$

Example 3.8.2: Let Σ_g be an orientable surface of genus $g \geq 0$, with fundamental group σ_g. Because $\Sigma_g = K(\sigma_g, 1)$ is a PD^2-complex of Euler characteristic $2-2g$,

$$ \beta_i(\Sigma_g) = \beta_i(\sigma_g) = \begin{cases} 2g-2, & \text{for } i = 1, \\ 0, & \text{else.} \end{cases} $$

Example 3.8.3: Let X be a finite PD^2-complex with infinite fundamental group. Then $b_1(X) > \beta_1(X) \geq 0$. In particular, $b_1(X) > 0$. Indeed, one has

$$ \chi(X) = 1 - b_1(X) + b_2(X) = -\beta_1(X) > -b_1(X). $$

(Actually, finite PD^2-complexes are homotopy equivalent to closed surfaces, see [12]; the result holds even for an arbitrary finitely dominated PD^2-complex, see [11].)

3.9. We can extend the definition of our ℓ_2-Betti numbers as in (2.6.3). If Y is a G-CW-complex with cocompact j-skeleton, we put

$$\beta_i(Y;G) := \beta_i(Y^j;G), \quad i < j.$$

Similarly the ℓ_2-Betti numbers $\beta_i(X)$ are defined for $i < j$ if X^j is a finite connected complex. Note that if X^{j+1} is finite too, $\beta_j(X)$ is defined and it is obvious from the definition that it satisfies

$$\beta_j(X) \leq \beta_j(X^j).$$

COROLLARY 3.9.1: *Let X be a connected CW-complex with finite j-skeleton and $(j-1)$-connected universal cover. Then for all $i < j$*

$$\beta_i(X) = \beta_i(\pi_1(X)).$$

Proof: The assumptions on X imply that one can construct a model Y for $K(\pi_1(X),1)$ by attaching cells of dimension $> j$ to X^j. Thus for all $i < j$

$$\beta_i(X) = \beta_i(X^j) = \beta_i(Y) = \beta_i(\pi_1(X)). \qquad \blacksquare$$

In particular, for any connected space X with finite 2-skeleton one has

$$\beta_1(X) = \beta_1(\pi_1(X)).$$

Remark 3.9.2: Let $Y \to Y/G = X$ be a regular covering of a compact oriented Riemannian manifold X. The L^2-harmonic forms ${}^{dR}\mathcal{H}^p(Y)$ form a Hilbert G-submodule of the *de Rham complex* of L^2-forms on Y and integration of forms over cochains (with respect to a suitable triangulation of Y) defines a morphism of Hilbert G-modules

$$\int : {}^{dR}\mathcal{H}^p(Y) \to {}^{red}H^p(Y).$$

Dodziuk proved in [4] that this is an isomorphism of $\mathbb{R}$-vector spaces. In particular the Betti numbers $\beta_p(Y;G)$ agree with the corresponding *de Rham* ℓ_2-Betti numbers $\dim_G {}^{dR}\mathcal{H}^p(Y)$.

Atiyah asked in [1] whether these numbers are rational (resp. integers, in the case of a torsion-free group G). The question is related to the conjectures below and also to a question concerning zero divisors in $\mathbb{Q}[G]$.

3.10. The following two statements are variations of what sometimes is referred to as **Atiyah's Conjecture**.

CONJECTURE A: *Let Y be a connected free cocompact G-CW-complex. Then all $\beta_i(Y;G)$ are rational numbers. If k is a positive integer such that the order of any finite subgroup of G divides k, then $k \cdot \beta_i(G)$ is an integer.*

Note that the group G in the conjecture is necessarily finitely generated, being a factor group of $\pi_1(Y/G)$.

CONJECTURE B: *Let $\phi\colon \mathbb{Z}[G]^m \to \mathbb{Z}[G]^n$ be a morphism of $\mathbb{Z}[G]$-modules, $\tilde{\phi}$ the induced bounded operator $\ell_2(G)^m \to \ell_2(G)^n$. Then $\dim_G \ker \tilde{\phi}$ is rational. If k is as above in **A** then $k \cdot \dim_G \ker \tilde{\phi}$ is an integer.*

Since the ϕ in the conjecture is induced by some $\bar{\phi}\colon \mathbb{Z}[\overline{G}]^m \to \mathbb{Z}[\overline{G}]^n$ for a suitable finitely generated subgroup $\overline{G} < G$, the conjecture holds if it holds for all finitely generated groups.

PROPOSITION 3.10.1: *For a finitely generated group G the two conjectures are equivalent.*

Proof: Assuming **A** and given ϕ as in **B**, it is easy to construct a Y as in **A** such that $\ker \tilde{\phi}$ is isomorphic to ${}^{red}H_3(Y)$. (Choose a surjection $F \to G$ with F finitely presented and choose a $K(F,1) = Z$ with finite 2-skeleton Z^2. Let $\vee_n S^2$ be a wedge of n two-spheres and define Y^2 to be the covering space of $Z^2 \bigvee (\vee_n S^2)$ associated with the kernel of $F \to G$. It is a free cocompact G-space; attach m free G-cells of dimension 3 to Y^2 to obtain Y with $\ker(K_3(Y) \to K_2(Y)) = \ker \phi$.) Conversely, assuming **B** one obtains **A** by observing that ${}^{red}H_i(Y) = \ker \tilde{\Delta}$, where the **combinatorial Laplacian** $\Delta\colon K_i(Y) \to K_i(Y)$ is defined in the obvious way (using the identification $K_j(Y) = \mathbb{Z}[G]^{\alpha_j}$, α_j the number of j-cells of Y/G). ∎

The **Zero Divisor Conjecture** states that for a torsion-free group G the group ring $\mathbb{Q}[G]$ does not contain any zero divisors $\neq 0$. Clearly the conjecture holds, if it holds for finitely generated groups. It is known to hold for a large class of groups (cf. [17]).

THEOREM 3.10.2: *The conjectures **A** and **B** imply the Zero Divisor Conjecture.*

Proof: Let G be a finitely generated torsion-free group and let $a, b \in \mathbb{Q}[G]$ with $a \neq 0$ and $ab = 0$; we need to show that $b = 0$. Consider the bounded G-equivariant operator

$$L_b\colon \ell_2 G \to \ell_2 G, \quad z \mapsto zb$$

and write M for its kernel. Since $a \in M$, $M \neq 0$ and therefore $0 < \dim_G M \leq 1$. Replacing b by nb for some $n > 0$ if necessary, we may assume $b \in \mathbb{Z}[G]$ so that $L_b = \tilde{\phi}$ for ϕ: $\mathbb{Z}[G] \to \mathbb{Z}[G]$ the right multiplication by b. Conjecture B now implies $\dim_G M = 1$, and therefore $M = \ell_2 G$ whence $b = 0$. ∎

4. Applications (deficiency, amenable groups)

Suppose that the group G possesses a presentation with g generators and r relators. Then obviously

$$g - r \leq \operatorname{rank}(G_{ab}) = b_1(G).$$

The maximal value $\operatorname{def}(G)$ of the differences $g-r$ over all finite presentations of G is called the **deficiency** of G. For example, a finite group G has $\operatorname{def}(G) \leq 0$. In the following we want to get some estimates for the deficiency of infinite groups.

If G is a finitely presented group with g generators and r relators we can construct a $K(G,1)$ with 2-skeleton $K(G,1)^2$ possessing 1 zero-cell, g one-cells and r two-cells. Taking Euler characteristics yields

$$r - g + 1 = b_2(K(G,1)^2) - b_1(K(G,1)^2) + 1.$$

But $b_1(K(G,1)^2) = b_1(G)$ and $b_2(K(G,1)^2) \geq b_2(G)$ so that in general

$$\operatorname{def}(G) \leq b_1(G) - b_2(G).$$

For ℓ_2-Betti numbers we get

$$\operatorname{def}(G) = 1 - \beta_0(G) + \beta_1(G) - \beta_2(K(G,1)^2).$$

Whence

THEOREM 4.1.1: *Let G be a finitely presented group. Then*

$$\operatorname{def}(G) \leq 1 + \beta_1(G).$$

In particular $\beta_1(G) = 0$ implies $\operatorname{def}(G) \leq 1$.

In case $K(G,1)$ has a finite 3-skeleton the Morse inequality (3.6.3, case $k = 2$) for the ℓ_2-Betti numbers of $K(G,1)^3$ yields

$$r - g + 1 \geq \beta_2(G) - \beta_1(G) + \beta_0(G).$$

Recall that a group G is of type F_n if and only if there is a $K(G,1)$ with finite n-skeleton.

THEOREM 4.1.2: *Let G be a group of type F_3. Then*

$$\operatorname{def}(G) \leq 1 + \beta_1(G) - \beta_2(G).$$

Example 4.1.3: If $G = *_n\mathbb{Z}$ is a free group of rank n, then using 3.8.1 we obtain

$$\operatorname{def}(*_n\mathbb{Z}) = n = 1 + \beta_1(*_n\mathbb{Z}).$$

Example 4.1.4: Let $G = \sigma_g$ be the fundamental group of an orientable surface of genus $g \geq 0$. The well known presentation for σ_g yields $\operatorname{def}(\sigma_g) \geq 2g - 1$ so that from 3.8.2 we obtain

$$\operatorname{def}(\sigma_g) = 2g - 1 = 1 + \beta_1(\sigma_g).$$

The next example is due to Lück [19].

Example 4.1.5: Let G be a finitely presented group possessing a finitely generated infinite normal subgroup N such that $\mathbb{Z} < G/N$. Then $\beta_1(G) = 0$ ([19], Theorem 0.7) and thus

$$\operatorname{def}(G) \leq 1.$$

The following variation of 4.1.2 is a consequence of 3.6.2.

THEOREM 4.1.6: *Let X be a connected finite CW-complex with fundamental group G, normal subgroup $N < G$ and $Q = G/N$. Let X_N^2 be the covering space of the 2-skeleton of X associated with N. Then*

$$\operatorname{def}(G) \leq 1 - \chi(X^2) \leq 1 + \beta_1(X_N^2; Q) - \beta_2(X_N^2; Q).$$

4.2. Let G be a group and B the space of bounded $\mathbb{R}$-valued functions on G. We consider B as a G-module by putting $(xf)(y) := f(yx)$ for all $x, y \in G$ and $f \in B$. A **mean** on G is a linear map $M\colon B \to \mathbb{R}$ such that for all $x \in G$ and $f \in B$

- $M(1) = 1$ $\quad$ ($1\colon G \to \mathbb{R}$ the constant function 1),
- $M(xf) = M(f)$,
- $f \geq 0 \Longrightarrow M(f) \geq 0$.

The following notion goes back to von Neumann.

Definition 4.2.1: A group G is called **amenable** if it admits a mean.

Example 4.2.2: A finite group G is amenable: it has a unique mean, given by

$$M(f) = \frac{1}{|G|} \sum_{x \in G} f(x).$$

It is known that the infinite cyclic group $\mathbb{Z}$ is amenable and that the class of amenable groups is

(i) extension closed, closed with respect to passing to subgroups and factor groups,

(ii) closed with respect to taking directed unions.

In particular all abelian and all solvable groups are amenable. The smallest class of groups containing all finite and all abelian groups, and which satisfies the closure properties (i) and (ii), is the class of *elementary amenable* groups. There do exist examples (even finitely presented ones) of amenable groups which are not elementary amenable (cf. [15]). On the other hand, it is a classical result that a group which contains a non-abelian free group cannot be amenable. For more information on amenability the reader is referred to [22].

4.3. Let Y be a free cocompact connected G-CW-complex. For results concerning the ℓ_2-Betti numbers $\beta_i(Y;G)$ for infinite amenable G, the following construction is most useful. Choose an (open) cell from each G-orbit of cells in Y and write $D \subset Y$ for their union and $\bar{D}$ for the closure of D in Y ($\bar{D}$ will not be a subcomplex of Y in general!). Since G is a factor group of $\pi_1(Y/G)$ it is countable: $G = \{g_\nu \mid \nu \in \mathbb{N}\}$. Construct an increasing family $\{Y_j\}_{j\in\mathbb{N}}$ of subspaces of Y as follows. Let $\{N_j\}$ be a strictly increasing sequence of natural numbers. Each Y_j is the union of N_j distinct translates $g_\nu D$, $\nu = 1, \dots, N_j$, $g_\nu \in G$ and $Y = \bigcup Y_j$. Let $\dot{N}_j$ be the the number of translates of $\bar{D}$ which meet the topological boundary $\dot{Y}_j$ of $Y_j \subset Y$. Using the *combinatorial Følner criterion* for amenability of G [14] it follows that the sequences $\{N_j, Y_j\}$ can be chosen such that $\dot{N}_j/N_j \to 0$ for $j \to \infty$ (cf. [5, 8]); we will call such a family a **Følner exhaustion.**

Recall that there is a canonical map

$$\mathrm{can}^i : {}^{red}\mathcal{H}^i(Y) \to H^i(Y;\mathbb{R})$$

induced by considering a harmonic ℓ_2-cocycle as an ordinary one. The following lemma is very useful.

LEMMA 4.3.1 (Cheeger–Gromov [3]): *Let Y be a connected free cocompact G-CW-complex, G an (infinite) amenable group. Then*

$$\mathrm{can}^i : {}^{red}\mathcal{H}^i(Y) \to H^i(Y;\mathbb{R})$$

is injective for all $i \geq 0$.

Proof: Choose a Følner exhaustion $\{N_j, Y_j \mid j \in \mathbb{N}\}$ for Y. View the kernel $\mathcal{K}$ of ${}^{red}\mathcal{H}^i(Y) \to H^i(Y;\mathbb{R})$ as a Hilbert G-submodule of $\mathcal{H}_i(Y) \subset C_i(Y) =: C_i$. Thus

$c \in \mathcal{K}$ is a harmonic chain $c = \sum c(\sigma)\sigma$ which, when considered as a cocycle $c(\ )$: $K_i(Y) \to \mathbb{R}$, is of the form $\delta^{i-1}b$ for some cochain b: $K_{i-1} \to \mathbb{R}$. Recall that C_i has a Hilbert basis which corresponds bijectively to the (open or closed) cells of Y and we will sometimes identify a cell of Y with the corresponding element in C_i. Let P: $C_i \to C_i$ stand for the orthogonal projection onto $\mathcal{K}$ and $\pi_{i,j} = \pi$: $C_i \to C_i$ the orthogonal projection onto the finite dimensional subspace spanned by the (open) i-cells which lie in Y_j. If R denotes the set of i-cells in D, then by definition

$$\dim_G \mathcal{K} = \sum_{\sigma\in R} \langle P(\sigma), \sigma\rangle.$$

Since $\pi \circ P$: $C_i \to C_i$ has a finite dimensional image, the ordinary trace

$$\operatorname{trace}_{\mathbb{R}} \pi \circ P = \sum_{\sigma\in R, x\in G} \langle (\pi P)(x\sigma), x\sigma\rangle$$

is defined. Now $\langle \pi P(x\sigma), x\sigma)\rangle = 0$ for cells $x\sigma$ not in Y_j; for cells $x\sigma \subset Y_j$

$$\langle \pi P(x\sigma), x\sigma\rangle = \langle P(x\sigma), x\sigma\rangle = \langle P(\sigma), \sigma\rangle,$$

implying

$$\operatorname{trace}_{\mathbb{R}} \pi P = N_j \sum_{\sigma\in R} \langle P(\sigma), \sigma\rangle = N_j \dim_G \mathcal{K}.$$

Since for any $c \in C_i$, $\|\pi P(c)\| \leq \|c\|$,

$$\operatorname{trace}_{\mathbb{R}} \pi P \leq \dim_{\mathbb{R}} \operatorname{im} \pi P = \dim_{\mathbb{R}} \pi(\mathcal{K})$$

whence

$$\dim_G \mathcal{K} \leq \frac{1}{N_j} \operatorname{trace}_{\mathbb{R}} \pi P \leq \frac{1}{N_j} \dim_{\mathbb{R}} \pi(\mathcal{K}).$$

To complete the proof, we need an estimate on $\dim_{\mathbb{R}} \pi(\mathcal{K})$.

Let σ be a cell in Y whose closure does not meet $\dot{Y}_j$. Then the same holds for the cells in $d_i\sigma$ since they lie in the closure of σ. For such a σ one has either $\pi\sigma = \sigma$ (if σ is in Y_j) and $\pi d\sigma = d\pi\sigma$; or $\pi\sigma = 0$ (if σ is not in Y_j) and $\pi d\sigma = 0 = d\pi\sigma$. Writing C'_* for the subspace of C_* having as Hilbert basis all cells σ in Y whose closure does not meet $\dot{Y}_j$ it follows for $c \in C'_i$

$$d_i \pi c = \pi d_i c.$$

As earlier we identify ℓ_2-chains with ℓ_2-cochains, yielding inclusions

$$K_*(Y) \otimes \mathbb{R} \subset C_* = C^* \subset K^*(Y) \otimes \mathbb{R}.$$

For $a \in K_*(Y) \otimes \mathbb{R}$ and $b \in K^*(Y) \otimes \mathbb{R}$ we write $b(a) \in \mathbb{R}$ for the usual evaluation of the cochain b on the chain a; if b happens to lie in C^* and we consider a as an ℓ_2-chain, the inner product $\langle a, b\rangle = b(a)$.

Consider now $c \in \mathcal{K} \cap C'_*$. It satisfies $d_i c = 0$ and $c = \delta^{i-1} b$ for some $b \in K^{i-1}(Y) \otimes \mathbb{R}$, and $d_i \pi c = \pi d_i c = 0$, yielding

$$\begin{aligned}\|\pi(c)\|^2 &= \langle \pi c, \pi c\rangle = \langle \pi c, c\rangle = \langle \pi c, \delta^{i-1} b\rangle \\ &= (\delta^{i-1} b)(\pi c) = b(d_i \pi c) = b(\pi d_i c) = 0.\end{aligned}$$

Whence

$$\mathcal{K} \cap C'_i \subset \ker\{\pi|_{\mathcal{K}} \colon \mathcal{K} \to \pi(\mathcal{K})\}$$

and therefore

$$\begin{aligned}\dim_{\mathbb{R}} \pi(\mathcal{K}) &\leq \dim_{\mathbb{R}} \mathcal{K}/\mathcal{K} \cap C'_i = \dim_{\mathbb{R}} (\mathcal{K} + C'_i)/C'_i \\ &\leq \dim_{\mathbb{R}} C_i/C'_i.\end{aligned}$$

The orthogonal complement $\dot{C}_*$ of C'_* in C_* has as Hilbert basis all cells in Y whose closure meets $\dot{Y}_j$. Now

$$\dim_{\mathbb{R}} C_i/C'_i = \dim_{\mathbb{R}} \dot{C}_i \leq \dot{N}_j \alpha(D)$$

where $\alpha(D)$ denotes the number of cells in D. It follows that

$$\dim_G \mathcal{K} \leq \frac{\dot{N}_j \alpha(D)}{N_j} \to 0 \qquad \text{as } j \to \infty,$$

and we conclude $\mathcal{K} = 0$. ∎

COROLLARY 4.3.2: *Let X be a finite connected CW-complex with amenable fundamental group G. Then $\beta_1(X) = \beta_1(G) = 0$.*

Proof: The universal cover Y of X satisfies $H^1(Y; \mathbb{R}) = 0$; thus the result follows from the Cheeger–Gromov Lemma. ∎

The corollary also shows that —as remarked earlier— a group G which contains a non-abelian free group cannot be amenable, because $\beta_1(\mathbb{Z} * \mathbb{Z}) = 1 \neq 0$.

Applying the Cheeger–Gromov Lemma to the m-skeleton of the universal cover of a $K(G, 1)$ one obtains the following *vanishing theorem* for ℓ_2-Betti numbers.

THEOREM 4.3.3: *Let G be a finitely presented infinite amenable group. Then $\beta_1(G) = 0$; if G is of type F_m then $\beta_i(G) = 0$ for $i < m$.*

COROLLARY 4.3.4: *Suppose G is an infinite amenable group admitting a finite $K(G,1)$. Then $\chi(G) = 0$.*

For the next application we need a general fact on Hilbert modules which is a generalization of 2.7.1.

LEMMA 4.3.5: *Let G be an infinite group and W a Hilbert G-module of finite dimension as an $\mathbb{R}$-vector space. Then $W = 0$.*

Proof: We may assume $W \subset (\ell_2 G)^n$ and, by induction, that $n = 1$. Let $\Pi \in N(G)$ be the orthogonal projection onto W so that

$$\dim_G W = \langle \Pi(1), 1\rangle = \langle \Pi(x), x\rangle \leq \|\Pi(x)\|, \quad \forall x \in G.$$

Choose an orthonormal basis $w_1, \ldots, w_n \in W$. Each w_i has the form $\sum_{g\in G} r_i(g)g$ with $\sum_{g\in G} r_i(g)^2 = 1$. Since G is infinite it is therefore possible to find for each $j \geq 0$ an element $x_j \in G$ such that $|r_i(x_j)| \leq 2^{-j}$, $i = 1, \ldots, n$. Note that

$$\langle \Pi(x_j), w_i\rangle = \langle x_j, \Pi(w_i)\rangle = \langle x_j, w_i\rangle = r_i(x_j).$$

With the x_j's above

$$\|\Pi(x_j)\|^2 = \sum_i r_i(x_j)^2 \leq n \cdot 2^{-2j} \longrightarrow 0 \quad \text{as } j \to \infty,$$

showing that $\dim_G W = 0$, whence $W = 0$. ∎

Combining the lemma with the Cheeger–Gromov Lemma yields

COROLLARY 4.3.6: *Let Y be a connected free cocompact G-CW-complex with (infinite) amenable G. Assume that the ordinary Betti number $b_i(Y) < \infty$ for some i. Then $\beta_i(Y; G) = 0$.*

COROLLARY 4.3.7: *Let G be a finitely presented quasi-amenable group (meaning that there exists a normal subgroup $N < G$ with $b_1(N) < \infty$ and G/N infinite amenable). Then* $\mathrm{def}(G) \leq 1$.

Proof: Choose a $K(G,1)$ with finite 2-skeleton X. Let Y be the covering space of X associated with N. From 4.1.6 we get

$$\mathrm{def}(G) \leq 1 + \beta_1(Y; G/N)$$

and the result follows since $\beta_1(Y; G/N) = 0$ for amenable G/N. ∎

For a free cocompact G-CW-complex Y there is a natural map

$$\mathrm{can}_i : H_i(Y;\mathbb{Z}) \to {}^{red}H_i(Y)$$

induced by considering an integral chain as an ℓ_2-chain; we can also view the map to be induced by the inclusion of chain complexes

$$K_i(Y) \to \ell_2(G) \otimes_{\mathbb{Z}[G]} K_i(Y) = C_i(Y).$$

LEMMA 4.3.8: *Let Y be an n-dimensional free cocompact G-CW-complex. Then*

$$\mathrm{can}_n : H_n(Y;\mathbb{Z}) \to {}^{red}H_n(Y)$$

is injective.

Proof: This follows immediately from the long exact homology sequence associated with

$$0 \to K_*(Y) \to C_*(Y) \to C_*(Y)/K_*(Y) \to 0,$$

and observing that because Y is n-dimensional, ${}^{red}H_n(Y) = H_n(C_*(Y))$. ∎

COROLLARY 4.3.9: *Let Y be a free cocompact $(n-1)$-connected G-CW-complex of dimension $n > 1$. Assume that one of the following conditions holds:*

- *the $\mathbb{R}$-vector space $H_n(Y;\mathbb{R})$ is finite dimensional,*
- *the ℓ_2-Betti number $\beta_n(Y;G) = 0$.*

Then Y is contractible.

Proof: Note that, because Y is n-dimensional, $H_n(Y;\mathbb{Z}) \subset H_n(Y;\mathbb{R})$. For the first case, use 4.3.5 and the previous lemma to conclude that $H_n(Y;\mathbb{Z}) = 0$. The Hurewicz Theorem then shows that Y is n-connected, thus contractible. Similarly for the second case. ∎

4.4. In this section we present a few applications concerning the *partial* Euler characteristic $\mathfrak{q}_m(G)$ of (amenable) groups as well as the *Hausmann–Weinberger Invariant* $\mathfrak{q}(G)$. For more results along these lines the reader is referred to [8, 9, 10]. Suppose G admits a $K(G,1)$ with finite m-skeleton. Put $X = K(G,1)^m$ and consider

$$(-1)^m\chi(X) = \sum_{i=0}^{m-1} b_i(G) + b_m(X) \geq \sum_{i=0}^{m-1} b_i(G).$$

Define

$$\mathfrak{q}_m(G) = \min\{(-1)^m \chi(X)\},$$

the minimum being taken over all possible choices of finite X as above.

In particular

- $\mathfrak{q}_1(G) = \min\{\text{number of generators of } G\}$,
- $\mathfrak{q}_2(G) = 1 - \operatorname{def}(G)$.

THEOREM 4.4.1: *Let G be an infinite amenable group of type F_m. Then $\mathfrak{q}_m(G) \geq 0$, and $\mathfrak{q}_m(G) = 0$ implies that the cohomology dimension of G over $\mathbb{Z}$ is $\leq m$.*

Proof: Let X be a finite model for $K(G,1)^m$. Since $\beta_i(X) = \beta_i(G) = 0$ for $i < m$ (cf. 4.3.3),

$$(-1)^m \chi(X) = \beta_m(X) \geq 0$$

and therefore $\mathfrak{q}_m(G) \geq 0$. If $\mathfrak{q}_m(G) = 0$ then we can choose $X = K(G,1)^m$ with $\beta_m(X) = 0$ and 4.3.9 applied to the universal cover of X implies that X is a $K(G,1)$ of dimension m (the case $m = 1$ cannot occur here, since for a non-trivial group $\mathfrak{q}_1 > 0$). ∎

The following definition goes back to Hausmann–Weinberger [16]. Let M be a closed (smooth) oriented 4-manifold. Then, since $b_1(M) = b_3(M) = b_1(G)$ for G the fundamental group of M,

$$\chi(M) = 2 - 2b_1(G) + b_2(M) \geq 2(1 - b_1(G)).$$

Put

$$\mathfrak{q}(G) = \min\{\chi(M)\}$$

where M runs over all manifolds as above, with $\pi_1(M) = G$. In a similar way, various authors have defined (see [10])

$$\mathfrak{p}(G) = \min\{\chi(M) + \sigma(M)\}$$

with M as before and $\sigma(M)$ the signature of M; the minimum exists because $|\sigma(M)| \leq b_2(M)$ so that

$$\chi(M) + \sigma(M) \geq 2(1 - b_1(G)).$$

Note also that $\mathfrak{p}(G) \leq \mathfrak{q}(G)$ as $\sigma(-M) = -\sigma(M)$.

It is well known that there exists for any finitely presented group G a closed smooth oriented manifold M with fundamental group G. The invariants $\mathfrak{p}(G)$ and $\mathfrak{q}(G)$ are therefore defined for any finitely presented group G.

Atiyah proved in [1] the ℓ_2-*Signature Theorem* which tells that ${}^{red}H^2(\tilde{M})$ splits into a Hilbert direct sum of two Hilbert G-modules with von Neumann dimensions $\beta_2^+(M)$ and $\beta_2^-(M)$ such that $\sigma(M) = \beta_2^+(M) - \beta_2^-(M)$. Expressing $\chi(M)$ in terms of ℓ_2-Betti numbers this leads in case of an infinite G to the formula

$$\chi(M) + \sigma(M) = -2\beta_1(G) + 2\beta_2^+(M).$$

Applying it to groups with vanishing first ℓ_2-Betti number we see that $\chi(M) + \sigma(M)$ is non-negative, which has interesting consequences, cf. [10]. In particular the case of an amenable G yields the following.

THEOREM 4.4.2: *Let G be a finitely presented amenable group. Then* $\mathrm{p}(G)$ *and* $\mathrm{q}(G)$ *are non-negative.*

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